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Inhomogeneous Inverse Differential Realization of Multimode *SU*(1, 1) Group

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The generators and irreducible representation coherent state of the multimode *SU*(1, 1) group in Bargmann space are constructed by using the inverse operators of the multimode bosonic harmonic oscillator, and the inhomogeneous inverse differential realization of the multimode *SU*(1, 1) group are derived.

The boson realization approach is very effective for studying the representation theory of groups, and the boson realization usually can be obtained from the creation and annihilation operators of the bosonic harmonic oscillator, as in the Jordan–Schwinger realization, etc. On the other hand, the inverse operator of the bosonic harmonic oscillator (Dirac, 1966) had been studied, and some new results have been given (Mehta and Roy, 1992; Fan, 1993, 1994). In consideration of the close relationship between the quantum mechanical quasi-accuration problem and the inhomogeneous differential realization of the Lie group (Turbiner and Ushveridze, 1987; Turbiner, 1988), the present paper studies the inhomogeneous inverse differential realization (a new kind of inhomogeneous differential realization) of the multimode *SU*(1, 1) group in Bargmann space by using the inverse operator of the multimode bosonic harmonic oscillator.

We first introduce four independent k-mode bosonic operators as follows:

$$A_k = a_1 a_2 \cdots a_k \left\{ \frac{n_1^a n_2^a \cdots n_k^a}{\min(n_1^a, n_2^a, \ldots, n_k^a)} \right\}^{-1/2} \tag{1}$$

$$B_k = b_1 b_2 \cdots b_k \left\{ \frac{n_1^b n_2^b \cdots n_k^b}{\min(n_1^b, n_2^b, \ldots, n_k^b)} \right\}^{-1/2} \tag{2}$$

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$$C_k = c_1c_2 \cdots c_k \left\{ \frac{n_1^c n_2^c \cdots n_k^c}{\min(n_1^c, n_2^c, \ldots, n_k^c)} \right\}^{-1/2} \tag{3}$$

$$D_k = d_1d_2 \cdots d_k \left\{ \frac{n_1^d n_2^d \cdots n_k^d}{\min(n_1^d, n_2^d, \ldots, n_k^d)} \right\}^{-1/2} \tag{4}$$

where

$$a_i^+ a_i = n_i^a, \qquad a_i a_i^+ = n_i^a + 1, \qquad [a_i, a_i^+] = 1 \tag{5}$$

$$[n_i^a, a_i] = -a_i, \qquad [n_i^a, a_i^+] = a_i^+ \tag{6}$$

$$b_i^+ b_i = n_i^b, \qquad b_i b_i^+ = n_i^b + 1, \qquad [b_i, b_i^+] = 1 \tag{7}$$

$$[n_i^b, b_i] = -b_i, \qquad [n_i^b, b_i^+] = b_i^+ \tag{8}$$

$$c_i^+ c_i = n_i^c, \qquad c_i c_i^+ = n_i^c + 1, \qquad [c_i, c_i^+] = 1 \tag{9}$$

$$[n_i^c, c_i] = -c_i, \qquad [n_i^c, c_i^+] = c_i^+ \tag{10}$$

$$d_i^+ d_i = n_i^d, \qquad d_i d_i^+ = n_i^d + 1, \qquad [d_i, d_i^+] = 1 \tag{11}$$

$$[n_i^d, d_i] = -d_i, \qquad [n_i^d, d_i^+] = d_i^+ \tag{12}$$

It is easy to check the following:

$$[A_k, A_k^+] = 1, \qquad [N_k^a, A_k] = -A_k, \qquad [N_k^a, A_k^+] = A_k^+ \tag{13}$$

$$[B_k, B_k^+] = 1, \qquad [N_k^b, B_k] = -B_k, \qquad [N_k^b, B_k^+] = B_k^+ \tag{14}$$

$$[C_k, C_k^+] = 1, \qquad [N_k^c, C_k] = -C_k, \qquad [N_k^c, C_k^+] = C_k^+ \tag{15}$$

$$[D_k, D_k^+] = 1, \qquad [N_k^d, D_k] = -D_k, \qquad [N_k^d, D_k^+] = D_k^+ \tag{16}$$

where

$$N_k^a = \min(n_1^a, n_2^a, \ldots, n_k^a) \tag{17}$$

$$N_k^b = \min(n_1^b, n_2^b, \ldots, n_k^b) \tag{18}$$

$$N_k^c = \min(n_1^c, n_2^c, \ldots, n_k^c) \tag{19}$$

$$N_k^d = \min(n_1^d, n_2^d, \ldots, n_k^d) \tag{20}$$

Equations (13)–(16) indicate that $\{A_k^+, A_k, N_k^a\}$, $\{B_k^+, B_k, N_k^b\}$, $\{C_k^+, C_k, N_k^c\}$, and $\{D_k^+, D_k, N_k^d\}$ can be regarded as k-mode bosonic harmonic oscillators, respectively.

Similar to Fan's works (Fan, 1993, 1994), the inverses of the k-mode bosonic harmonic oscillator creation and annihilation operators can be obtained from their actions on the number states $|n, n, \ldots\rangle$ (where $|n, n, \ldots\rangle = |n\rangle_1 |n\rangle_2 \cdots |n\rangle_k$):

$$A_k^{-1}|n, n, \ldots\rangle = \frac{1}{\sqrt{n+1}} |n+1, n+1, \ldots\rangle \tag{21}$$

$$(A_k^+)^{-1}|n, n, \ldots\rangle = \begin{cases} \frac{1}{\sqrt{n}} |n-1, n-1, \ldots\rangle, & n \neq 0 \\ 0, & n = 0 \end{cases} \tag{22}$$

$$B_k^{-1}|n, n, \ldots\rangle = \frac{1}{\sqrt{n+1}} |n+1, n+1, \ldots\rangle \tag{23}$$

$$(B_k^+)^{-1}|n, n, \ldots\rangle = \begin{cases} \frac{1}{\sqrt{n}} |n-1, n-1, \ldots\rangle, & n \neq 0 \\ 0, & n = 0 \end{cases} \tag{24}$$

$$C_k^{-1}|n, n, \ldots\rangle = \frac{1}{\sqrt{n+1}} |n+1, n+1, \ldots\rangle \tag{25}$$

$$(C_k^+)^{-1}|n, n, \ldots\rangle = \begin{cases} \frac{1}{\sqrt{n}} |n-1, n-1, \ldots\rangle, & n \neq 0 \\ 0, & n = 0 \end{cases} \tag{26}$$

$$D_k^{-1}|n, n, \ldots\rangle = \frac{1}{\sqrt{n+1}} |n+1, n+1, \ldots\rangle \tag{27}$$

$$(D_k^+)^{-1}|n, n, \ldots\rangle = \begin{cases} \frac{1}{\sqrt{n}} |n-1, n-1, \ldots\rangle, & n \neq 0 \\ 0, & n = 0 \end{cases} \tag{28}$$

which indicate that

$$A_k^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n+1, n+1, \ldots\rangle\langle n, n, \ldots| \tag{29}$$

$$\begin{aligned} (A_k^+)^{-1} &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n, n, \ldots\rangle\langle n+1, n+1, \ldots| \\ &= (A_k^{-1})^+ \end{aligned} \tag{30}$$

$$B_k^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n+1, n+1, \ldots\rangle\langle n, n, \ldots| \tag{31}$$

$$(B_k^+)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \left| n, n, \ldots \rangle\langle n+1, n+1, \ldots \right| = (B_k^{-1})^+ \tag{32}$$

$$C_k^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \left| n+1, n+1, \ldots \rangle\langle n, n, \ldots \right| \tag{33}$$

$$(C_k^+)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \left| n, n, \ldots \rangle\langle n+1, n+1, \ldots \right| = (C_k^{-1})^+ \tag{34}$$

$$D_k^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \left| n+1, n+1, \ldots \rangle\langle n, n, \ldots \right| \tag{35}$$

$$(D_k^+)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \left| n, n, \ldots \rangle\langle n+1, n+1, \ldots \right| = (D_k^{-1})^+ \tag{36}$$

We have the following noncommutation relations between A_k and A_k^{-1}, B_k and B_k^{-1}, C_k and C_k^{-1}, and D_k and D_k^{-1}:

$$A_k A_k^{-1} = (A_k^+)^{-1} A_k^+ = 1 \tag{37}$$

$$A_k^{-1} A_k = A_k^+ (A_k^+)^{-1} = 1 - \left| 0, 0, \ldots \rangle\langle 0, 0, \ldots \right| \tag{38}$$

$$B_k B_k^{-1} = (B_k^+)^{-1} B_k^+ = 1 \tag{39}$$

$$B_k^{-1} B_k = B_k^+ (B_k^+)^{-1} = 1 - \left| 0, 0, \ldots \rangle\langle 0, 0, \ldots \right| \tag{40}$$

$$C_k C_k^{-1} = (C_k^+)^{-1} C_k^+ = 1 \tag{41}$$

$$C_k^{-1} C_k = C_k^+ (C_k^+)^{-1} = 1 - \left| 0, 0, \ldots \rangle\langle 0, 0, \ldots \right| \tag{42}$$

$$D_k D_k^{-1} = (D_k^+)^{-1} D_k^+ = 1 \tag{43}$$

$$D_k^{-1} D_k = D_k^+ (D_k^+)^{-1} = 1 - \left| 0, 0, \ldots \rangle\langle 0, 0, \ldots \right| \tag{44}$$

where A_k^{-1}, B_k^{-1}, C_k^{-1}, and D_k^{-1} stand for the right inverses of A_k, B_k, C_k, and D_k, and $(A_k^+)^{-1}$, $(B_k^+)^{-1}$, $(C_k^+)^{-1}$, and $(D_k^+)^{-1}$ stand for the left inverses of A_k^+, B_k^+, C_k^+, and D_k^+, respectively.

Similar to the single-mode $SU(1, 1)$ group (Holman *et al.*, 1966; Bargmann, 1947), the k-mode $SU(1, 1)$ group also has three types of unitary irreducible representations: (a) a positive discrete series, (b) a negative discrete series, and (c) a continuous series which we do not consider here. The generators of the k-mode $SU(1, 1)$ group can be obtained from a Jordan–Schwinger realization in terms of the combination of inverse operators:

$$(L_+^a)^{-1} = (A_k^+)^{-1}(B_k^+)^{-1}, \qquad (L_-^a)^{-1} = A_k^{-1}B_k^{-1} \tag{45}$$

$$(L_0^a)^{-1} = \tfrac{1}{2}[(N_{k,A}^a)^{-1}(N_{k,B}^a)^{-1} - (N_{k,A}^a + 1)^{-1}(N_{k,B}^a + 1)^{-1}] \tag{46}$$

$$(L_+^b)^{-1} = D_k^{-1}C_k^{-1}, \qquad (L_-^b)^{-1} = (C_k^+)^{-1}(D_k^+)^{-1} \tag{47}$$

$$(L_0^b)^{-1} = \frac{-1}{2}[(N_{k,C}^b)^{-1}(N_{k,D}^b)^{-1} - (N_{k,C}^b + 1)^{-1}(N_{k,D}^b + 1)^{-1} \tag{48}$$

where

$$(N_{k,A}^a)^{-1} = A_k^{-1}(A_k^+)^{-1}, \qquad (N_{k,A}^a + 1)^{-1} = (A_k^+)^{-1}A_k^{-1} \tag{49}$$

$$(N_{k,B}^a)^{-1} = B_k^{-1}(B_k^+)^{-1}, \qquad (N_{k,B}^a + 1)^{-1} = (B_k^+)^{-1}B_k^{-1} \tag{50}$$

$$(N_{k,C}^b)^{-1} = C_k^{-1}(C_k^+)^{-1}, \qquad (N_{k,C}^b + 1)^{-1} = (C_k^+)^{-1}C_k^{-1} \tag{51}$$

$$(N_{k,D}^b)^{-1} = D_k^{-1}(D_k^+)^{-1}, \qquad (N_{k,D}^b + 1)^{-1} = (D_k^+)^{-1}D_k^{-1} \tag{52}$$

which satisfy the relations

$$(N_{k,A}^a)^{-1}|n, n, \ldots\rangle = \frac{1}{n}|n, n, \ldots\rangle,$$

$$(N_{k,A}^a + 1)^{-1}|n, n, \ldots\rangle = \frac{1}{n+1}|n + 1, n + 1, \ldots\rangle \tag{53}$$

$$(N_{k,B}^a)^{-1}|n, n, \ldots\rangle = \frac{1}{n}|n, n, \ldots\rangle,$$

$$(N_{k,B}^a + 1)^{-1}|n, n, \ldots\rangle = \frac{1}{n+1}|n + 1, n + 1, \ldots\rangle \tag{54}$$

$$(N_{k,C}^b)^{-1}|n, n, \ldots\rangle = \frac{1}{n}|n, n, \ldots\rangle,$$

$$(N_{k,C}^b + 1)^{-1}|n, n, \ldots\rangle = \frac{1}{n+1}|n + 1, n + 1, \ldots\rangle \tag{55}$$

$$(N_{k,D}^b)^{-1}|n, n, \ldots\rangle = \frac{1}{n}|n, n, \ldots\rangle,$$

$$(N_{k,D}^b + 1)^{-1}|n, n, \ldots\rangle = \frac{1}{n+1}|n + 1, n + 1, \ldots\rangle \tag{56}$$

It is easy to find that the following commutative relation holds:

$$(L_+^{a(b)})^{-1}(L_-^{a(b)})^{-1} - (L_-^{a(b)})^{-1}(L_+^{a(b)})^{-1} = -2(L_0^{a(b)})^{-1} \tag{57}$$

The two discrete unitary irreducible representations $|k, r\rangle^a$ and $|k, r\rangle^b$ of the k-mode $SU(1, 1)$ group are, respectively,

$$|k, r\rangle^a = |r - k - 1\rangle_1^a \otimes |r + k\rangle_2^a \qquad (r \geq -k > 0) \tag{58}$$

$$|k, r\rangle^b = |-r - k - 1\rangle_1^b \otimes |-r + k\rangle_2^b \qquad (r \leq k < 0) \tag{59}$$

These irreducible representations are infinite dimensional and depend on the quantum numbers $k = -1/2, -1, \ldots$. The action of the k-mode $SU(1, 1)$ group generators on the elements of the irreducible representations (58) and (59) is given by

$$(L_+^a)^{-1}|k, r\rangle^a = \frac{1}{\sqrt{(r - k - 1)(r + k)}}|k, r - 1\rangle^a \tag{60}$$

$$(L_-^a)^{-1}|k, r\rangle^a = \frac{1}{\sqrt{(r - k)(r + k + 1)}}|k, r + 1\rangle^a \tag{61}$$

$$(L_0^a)^{-1}|k, r\rangle^a = \frac{r}{(r + k + 1)(r + k)(r - k)(r - k - 1)}|k, r\rangle^a \tag{62}$$

$$(L_+^b)^{-1}|k, r\rangle^b = \frac{1}{\sqrt{(-r - k)(-r + k + 1)}}|k, r - 1\rangle^b \tag{63}$$

$$(L_-^b)^{-1}|k, r\rangle^b = \frac{1}{\sqrt{(-r + k)(-r - k - 1)}}|k, r + 1\rangle^b \tag{64}$$

$$(L_0^b)^{-1}|k, r\rangle^b = \frac{-r}{(r + k + 1)(r + k)(r - k)(r - k - 1)}|k, r\rangle^b \tag{65}$$

The coherent states of the irreducible representation for the k-mode $SU(1, 1)$ group corresponding to (a) the positive discrete series and (b) the negative discrete series are, respectively,

$$\begin{aligned}|kz\rangle^a &= e^{z(L_-^a)^{-1}}|k, -k\rangle^a \\ &= \sum_{r=-k}^{\infty} \frac{1}{(k + r)!}\sqrt{\frac{(-2k - 1)!}{(k + r)!\,(r - k - 1)!}}\, z^{k+r}|k, r\rangle^a \end{aligned} \tag{66}$$

$$\begin{aligned}|kz\rangle^b &= e^{z(L_+^b)^{-1}}|k, k\rangle^b \\ &= \sum_{r=k}^{-\infty} \frac{1}{(k - r)!}\sqrt{\frac{(-2k - 1)!}{(k - r)!\,(-r - k - 1)!}}\, z^{k-r}|k, r\rangle^b \end{aligned} \tag{67}$$

Their normalization coefficients are

$$A_k^a(|z|^2) = \sum_{r=-k}^{\infty} \frac{(-2k-1)!}{[(k+r)!]^3(r-k-1)!}(|z|^2)^{k+r} \tag{68}$$

$$B_k^b(|z|^2) = \sum_{r=k}^{-\infty} \frac{(-2k-1)!}{[(k-r)!]^3(-r-k-1)!}(|z|^2)^{k-r} \tag{69}$$

In the following discussion, the method we use is based on the idea proposed by Yu *et al.* (1997a, b). In order to construct the completeness relations of the quantum states $|kz)^a$ and $|kz)^b$, we define $P^a(k+r, z)$ and $P^b(k-r, z)$ to be the observable probabilities for $|k, r\rangle^a$ in state $|kz)^a$ and for $|k, r\rangle^b$ in state $|kz)^b$, respectively:

$$\begin{aligned} P^a(k+r, z) &= |{}^a\langle k, r|kz)^a|^2 \\ &= \frac{(-2k-1)!}{[(k+r)!]^3(r-k-1)!}(|z|^2)^{k+r} \end{aligned} \tag{70}$$

$$\begin{aligned} P^b(k-r, z) &= |{}^b\langle k, r|kz)^b|^2 \\ &= \frac{(-2k-1)!}{[(k-r)!]^3(-r-k-1)!}(|z|^2)^{k-r} \end{aligned} \tag{71}$$

Setting $P^a(k+r) = \int P^a(k+r, z)\, dz^2$ and $P^b(k-r) = \int P^b(k-r, z)\, dz^2$ and letting ρ^a and ρ^b represent the density matrixes of states $|k, r\rangle^a$ and $|k, r\rangle^b$, respectively, we have

$$\rho^a = \sum_{r=-k}^{\infty} P^a(k+r)|k, r\rangle^{aa}\langle k, r|, \qquad \rho^b = \sum_{r=k}^{-\infty} P^b(k-r)|k, r\rangle^{bb}\langle k, r| \tag{72}$$

The completeness relations of the states $|kz)^a$ and $|kz)^b$ can be written as

$$\frac{1}{\pi}(\rho^a)^{-1}\int \frac{|kz)^{aa}(kz|}{A_k^a(|z|^2)}dz^2 = 1, \qquad \frac{1}{\pi}(\rho^b)^{-1}\int \frac{|kz)^{bb}(kz|}{B_k^b(|z|^2)}dz^2 = 1 \tag{73}$$

We now define the Bargmann representations of the bases $|k, r\rangle^a$ and $|k, r\rangle^b$ for the irreducible representations as follows:

$$\begin{aligned} f_{kr}^a(z) &= {}^a(k\bar{z}|k, r\rangle^a \\ &= \frac{1}{(k+r)!}\sqrt{\frac{(-2k-1)!}{(k+r)!\,(r-k-1)!}}\, z^{k+r} \end{aligned} \tag{74}$$

$$f_{kr}^{b}(z) = {}^{b}(k\bar{z}|k, r\rangle^{b}$$
$$= \frac{1}{(k-r)!}\sqrt{\frac{(-2k-1)!}{(k-r)!\,(-r-k-1)!}}\, z^{k-r} \tag{75}$$

Defining two state vectors in the space of the irreducible representations

$$|\psi\rangle^{a} = \sum_{r=-k}^{\infty} C_{r}^{a}|k, r\rangle^{a}, \qquad |\psi\rangle^{b} = \sum_{r=k}^{-\infty} C_{r}^{b}|k, r\rangle^{b} \tag{76}$$

we have

$${}^{a}(k\bar{z}|(L_{-}^{a})^{-1}|\psi\rangle^{a}$$
$$= \sum_{r=-k}^{\infty} C_{r}^{a}\, {}^{a}(k\bar{z}|(L_{-}^{a})^{-1}|k, r\rangle^{a}$$
$$= \sum_{r=-k}^{\infty} \frac{C_{r}^{a}}{(k+r+1)^{2}(k+r)!\,(r-k)} \times \sqrt{\frac{(-2k-1)!}{(k+r)!\,(r-k-1)!}}\, z^{k+r+1} \tag{77}$$

$${}^{b}(k\bar{z}|(L_{+}^{b})^{-1}|\psi\rangle^{b}$$
$$= \sum_{r=k}^{-\infty} C_{r}^{b}\, {}^{b}(k\bar{z}|(L_{+}^{b})^{-1}|k, r\rangle^{b}$$
$$= \sum_{r=k}^{-\infty} \frac{C_{r}^{b}}{(k-r+1)!\,(k-r+1)(-r-k)} \times \sqrt{\frac{(-2k-1)!}{(k-r)!\,(-r-k-1)!}}\, z^{k-r+1} \tag{78}$$

On the other hand, we have

$$\frac{1}{(-2k + z\, d/dz)(1/z)(z\, d/dz)^{2}}\, {}^{a}(k\bar{z}|\psi\rangle^{a}$$
$$= \sum_{r=-k}^{\infty} \frac{C_{r}^{a}}{(k+r+1)^{2}(k+r)!\,(r-k)} \times \sqrt{\frac{(-2k-1)!}{(k+r)!\,(r-k-1)!}}\, z^{k+r+1} \tag{79}$$

$$
\begin{aligned}
&\frac{1}{(-2k + z\,d/dz)(1/z)(z\,d/dz)^2}\; {}^b(k\bar{z}|\psi\rangle^b \\
&= \sum_{r=k}^{-\infty} \frac{C_r^b}{(k-r+1)!\,(k-r+1)(-r-k)} \\
&\quad \times \sqrt{\frac{(-2k-1)!}{(k-r)!\,(-r-k-1)!}}\; z^{k-r+1}
\end{aligned} \tag{80}
$$

From equations (77)–(80), the inhomogeneous inverse differential realizations of $(L_-^a)^{-1}$ and $(L_+^b)^{-1}$ in the Bargmann space are

$$
B_k^a((L_-^a)^{-1}) = \frac{1}{(-2k + z\,d/dz)(1/z)(z\,d/dz)^2} \tag{81}
$$

$$
B_k^b((L_+^b)^{-1}) = \frac{1}{(-2k + z\,d/dz)(1/z)(z\,d/dz)^2} \tag{82}
$$

Similarly, we have

$$
\begin{aligned}
{}^a(k\bar{z}|(L_+^a)^{-1}|\psi\rangle^a &= \sum_{r=-k}^{\infty} \frac{C_r^a}{(k+r-1)!} \sqrt{\frac{(-2k-1)!}{(k+r)!\,(r-k-1)!}}\; z^{k+r-1} \\
&= \frac{d}{dz}\; {}^a(k\bar{z}|\psi\rangle^a
\end{aligned} \tag{83}
$$

$$
\begin{aligned}
{}^b(k\bar{z}|(L_-^b)^{-1}|\psi\rangle^b &= \sum_{r=k}^{-\infty} \frac{C_r^b}{(k-r-1)!} \sqrt{\frac{(-2k-1)!}{(k-r)!\,(-r-k-1)!}}\; z^{k-r-1} \\
&= \frac{d}{dz}\; {}^b(k\bar{z}|\psi\rangle^b
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
&{}^a(k\bar{z}|(L_0^a)^{-1}|\psi\rangle^a \\
&= \sum_{r=-k}^{\infty} \frac{rC_r^a}{(r+k+1)!\,(r+k)(r-k)(r-k-1)} \\
&\quad \times \sqrt{\frac{(-2k-1)!}{(k+r)!\,(r-k-1)!}}\; z^{k+r} \\
&= \left(z\frac{d}{dz} - k\right) \frac{1}{z\left(-2k + z\frac{d}{dz}\right)\left(\frac{1}{z}\right)\left(-2k + z\frac{d}{dz}\right)\left(\frac{1}{z}\right)\left(z\frac{d}{dz}\right)(z)\left(z\frac{d}{dz}\right)}\; {}^a(k\bar{z}|\psi\rangle^a
\end{aligned} \tag{85}
$$

$$
\begin{aligned}
&{}^{b}\langle k\bar{z}|(L_0^b)^{-1}|\psi\rangle^{b} \\
&= \sum_{r=k}^{-\infty} \frac{-rC_r^b}{(k-r)!\,(r+k+1)(r+k)(r-k)(r-k-1)} \\
&\quad \times \sqrt{\frac{(-2k-1)!}{(k-r)!\,(-r-k-1)!}}\, z^{k-r} \\
&= \left(z\frac{d}{dz}-k\right)\frac{1}{z\left(2k-z\frac{d}{dz}\right)\left(\frac{1}{z}\right)\left(2k-z\frac{d}{dz}\right)\left(\frac{1}{z}\right)\left(-z\frac{d}{dz}\right)(z^2)\left(-\frac{d}{dz}\right)}\,{}^{b}\langle k\bar{z}|\psi\rangle^{b}
\end{aligned}
\tag{86}
$$

Therefore we obtain the following inhomogeneous inverse differential realizations of $(L_+^a)^{-1}$, $(L_0^a)^{-1}$ and $(L_-^b)^{-1}$, $(L_0^b)^{-1}$ in the Bargmann space, respectively:

$$
B_k^a((L_+^a)^{-1}) = \frac{d}{dz} \tag{87}
$$

$$
B_k^a((L_0^a)^{-1}) = \left(z\frac{d}{dz}-k\right) \times \frac{1}{z\left(-2k+z\frac{d}{dz}\right)\left(\frac{1}{z}\right)\left(-2k+z\frac{d}{dz}\right)\left(\frac{1}{z}\right)\left(z\frac{d}{dz}\right)\left(z\right)\left(z\frac{d}{dz}\right)} \tag{88}
$$

$$
B_k^b((L_-^b)^{-1}) = \frac{d}{dz} \tag{89}
$$

$$
B_k^b((L_0^b)^{-1}) = \left(z\frac{d}{dz}-k\right) \times \frac{1}{z\left(2k-z\frac{d}{dz}\right)\left(\frac{1}{z}\right)\left(2k-z\frac{d}{dz}\right)\left(\frac{1}{z}\right)\left(-z\frac{d}{dz}\right)\left(z^2\right)\left(-\frac{d}{dz}\right)} \tag{90}
$$

In the above calculations of equations (79), (80), (85) and (86), we have used the inverse derivative formula (Ye, 1979). Finally, we point out that equations (81), (82), and (87)–(90) are the inhomogeneous inverse differential realizations of the k-mode $SU(1,\,1)$ group.

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